Topology Midterm Exam

March 26, 2020

1. Let X be a topological space and consider $Y = X \times X$ equipped with the topology generated by the basis

$$\mathcal{B} = \{U \times U \,|\, U \subset X \text{ is open}\}.$$

   - Show that $\mathcal{B}$ is a basis, and that the topology on Y is coarser than the product topology. **(1 pt)**
   - Show that the diagonal map $\Delta : X \to Y$ defined by $\Delta(x) = (x, x)$ is continuous. **(1 pt)**
   - If X is Hausdorff, does it follow that Y is Hausdorff? Give a proof or a counterexample. **(1 pt)**

2. Decide whether the following statements are true or false. If true give a proof, if false a counterexample.

   - The real line $\mathbb{R}$ with Euclidean topology is homeomorphic to the real line with cofinite topology. **(1 pt)**
   - $[0, 1]$ is homeomorphic to $[0, 1)$. **(1 pt)**
   - There exists an embedding of S^1 into $\mathbb{R}$. **(1 pt)**
   - There exists an embedding of $S^1 \times \mathbb{R}$ into $\mathbb{R}^2$. **(1 pt)**

3. Let $f : X \to Y$ be a continuous map between topological spaces. The *mapping cylinder* M_f of f is defined as the topological quotient

$$M_f := (X \times I) \sqcup Y / \sim,$$

   where the equivalence relation $\sim$ is given by $(x, 0) \sim f(x)$ for all $x \in X$.

   - Show that if Z is another topological space and $g : M_f \to Z$ is an arbitrary map, then g is continuous if and only if the induced maps $X \times I \to Z$ and $Y \to Z$ are continuous. **(2 pts)**

- Show that if $f : X \to Y$ is a map between Hausdorff spaces, then M_f is a Hausdorff space. **(1 pt)**

4. In $\mathbb{R}^3$ consider the surface

$$T^2_{a,b} = \{(x, y, z) \in \mathbb{R}^3 \, : \, (\sqrt{x^2 + y^2} - b)^2 + z^2 = a^2\}.$$

obtained by rotating around the z-axis the circle in the (x, z)-plane with centre in $(b, 0, 0)$ and radius a. Show that the map $f : S^1 \times S^1 \to T^2_{a,b}$ defined by

$$f((\cos u, \sin u), (\cos v, \sin v)) = ((b + a \cos u) \cos v, (b + a \cos u) \sin v, a \sin u)$$

is a homeomorphism. **(2 pts)**

Grading

$$\text{Final mark} = 1 + \frac{3}{4}(\#\text{points})$$