

Give your answers in English.
It is not allowed (nor useful) to use calculators.
Good luck!

1. (Arbitrage arguments)

Consider a forward contract corresponding to an agreement to buy an asset on a specified future date T for a specified price K . Assume that at the time the contract is signed the value of the asset is S_0 and let r be the risk-free interest rate. Use an arbitrage argument to obtain the fair price of the forward.

2. (Discrete-time martingales)

In this exercise time is discrete, (\mathcal{F}_n) is a given filtration. Consider a discrete time process X such that $X_{n+1} = f_n(X_n)$, where f_n are deterministic functions. Show that if X is a martingale, then it is constant, i.e., $X_n = X_0$ almost surely, for all n .

3. (Random walk)

Let S be the simple random walk. This means that $S_0 = 0$ and $S_n = X_1 + \dots + X_n$, where the X_i 's are independent and $\mathbb{P}_p(X_i = 1) = 1 - \mathbb{P}_p(X_i = -1) = p$. Let (\mathcal{F}_i) be the natural filtration of S .

- Explain why S is a Markov process.
- Give an expression for $\mathbb{E}_p(S_{i+1} | X_1, \dots, X_i)$.
- Show that S is a martingale with respect to (\mathcal{F}_i) if and only if $p = 1/2$.

4. (Brownian motion)

Let W be a Brownian motion and a a positive real number. Denote the natural filtration of W by (\mathcal{F}_t) .

- Show that the process X defined by $X_t = a^{-1/2}W_{at}$ is a Brownian motion with respect to its natural filtration.
- Use Ito's formula to show that the process Y defined by $Y_t = W_{at}^2 - at$ is a martingale.

5. (Stochastic calculus)

Let W be a Brownian motion and denote the natural filtration of W by (\mathcal{F}_t) .

- Compute $\int_0^t W_s dW_s$.

(b) Compute $\mathbb{E}(\int_0^t W_s dW_s)^2$.

(c) Let S be the process defined by $S_t = \exp(\sigma W_t + \alpha + \beta t)$. For which choice of parameters α , β and σ is S a martingale?

6. (Black-Scholes)

Let S and B denote the stock and bond price processes in a Black-Scholes market. Assume that the risk-free interest rate is zero. Consider a portfolio (φ, ψ) , where φ_t is the number of stocks held at time t and ψ_t the number of bonds. Suppose that $\varphi_t = t$ and $\psi_0 = 0$. Determine ψ_t in such a way that the portfolio becomes self-financing.

7. (Silly stock market model)

Consider a market in which a stock is traded with price process $S_t = W_t^2 - t$, where W is a Brownian motion under the “real-world” probability measure \mathbb{P} , and with a bank with zero interest. Let $T > 0$ and let $C = f(S_T)$ be a European claim with value V_t at time t .

(a) What is the martingale measure \mathbb{Q} ?

(b) Give an integral expression for the price V_0 of the derivative at time 0.

Norming:

1:	3	2:	3	3(a):	1	4(a):	3	5(a):	5	6:	4	7(a):	4
				3(b):	1	4(b):	2	5(b):	1			7(b):	4
				3(c):	1			5(c):	4				

Grade = (total+4)/4