Faculteit der Exacte Wetenschappen	Partiële Differentiaalvergelijkingen (401023)
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No calculators. You can bring your book and notes. Explain what you do. Do **exercises 1,2,3** (3 points each). Grade: total score plus 1.

1. Consider for $u = u(t,x)$ the first order partial differential equation (PDE)

$$u_t + c(x,u)u_x = h(x,u),$$

with initial condition at $t = 0$ given by

$$u(0,x) = f(x) > 0.$$

Here $x \in \mathbb{R}$ and $t \geq 0$. We assume that $(x,u) \to c(x,u)$ and $(x,u) \to h(x,u)$ define smooth functions on $\mathbb{R}^2$, and that $f : \mathbb{R} \to \mathbb{R}$ is also smooth.

(a) Let $x = X(t)$ be a smooth curve in the t,x-plane and assume that $u = u(t,x)$ is a smooth solution of the (PDE). Let $U(t) = u(t, X(t))$. Derive an equation of the form $\dot{X} = ..$ in terms of X and U that leads to an equation of the form $\dot{U} = ..$ in terms of X and U.

(b) Take $c(x,u) = c(x) = 1 - x^2$. The differential equation for $X(t)$ then decouples from the equation for $U(t)$. Its solutions define all the characteristic curves in the t,x-plane. Determine the general solution of the differential equation for $x = X(t)$ in the range $-1 < x < 1$.

(c) If the characteristic curve through a given point (t,x) intersects the vertical axis in the t,x-plane we denote the point of intersection by $(0,k)$. Derive an equation for k in terms of the coordinates t and x of the given point if $-1 < x < 1$.

(d) Take again $c(x,u) = c(x) = 1 - x^2$ and $h(x,u) \equiv 0$. Give a second equation for the value of $u = u(t,x)$ in the same given point (t,x) which involves k and $f(k)$. Determine $u = u(t,x)$ when $t > 0$ and $-1 < x < 1$.

(e) Now take $c(x,u) = c(x) = 1 - x^2$ and $h = h(u) = -u^2$. Then the second equation in Part 1d has to be replaced by another equation for the value of $u = u(t,x)$ in the same given point (t,x). Determine again $u = u(t,x)$ when $t > 0$ and $-1 < x < 1$.

(f) For which (t,x) with $t > 0$ are the solution formula's you found also valid?

2. Let $\beta \in \mathbb{R}$. Consider for $u = u(t,x)$ the equation

$$u_t = u_{xx}$$

with $0 < x < 1$. Given boundary conditions

$$u(t,0) = 0 = u_x(t,1) + \beta u(t,1),$$

the PDE has solutions of the form $u(t,x) = T(t)X(x)$. For which β do such solutions exist with $T(t) \to \infty$?

3. Let $f : \mathbb{R} \to \mathbb{R}$ be an odd and 2π-periodic piecewise smooth function. The Fourier series of f is given by

$$\sum_{n=1}^{\infty} b_n \sin nx.$$

(a) To make sure you use the right formula's for the Fourier coefficients: derive the integral formula's for b_n in the case that

$$f(x) = \sum_{n=1}^{N} b_n \sin nx$$

for some integer $N > 0$.

(b) Use the case that $f(x) = \pi - x$ for $0 < x < \pi$ to compute

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

in terms of π by using the explicit values of b_n. Explain why your answer is correct.

(c) For the same f, let

$$s_N(x) = \sum_{n=1}^{N} b_n \sin nx.$$

Show that $s_N'(x) + 1$ is a multiple of

$$D_N(x) = \frac{\sin(N + \frac{1}{2})x}{\sin \frac{x}{2}}$$

by using the complex formula's for $\cos nx$ and the values of b_n.