Resit Exam Operations Research

Date: June 2, 2021
Time: 12:15 - 14:30

Points per exercise:

- Exercise 1 has a total of 30 points (a(10), b(5), c(5) and d(10)).
- Exercise 2 has a total of 15 points.
- Exercise 3 has a total of 15 points (a(5) and b(10)).
- Exercise 4 has a total of 10 points.
- Exercise 5 has a total of 20 points (a(10) and b(10)).

Thus in total 90 points can be obtained. The exam grade is determined as follows:

$$\text{Exam grade } = 1 + \frac{\text{total number of obtained points}}{10}.$$

- Calculator is allowed.
- This exam consists of 5 pages, including this one.
- The duration of this exam is **2 hours and 15 minutes**. When you finish working you notify the host of your meeting. Within 15 minutes after you finish working you have to upload the scanned pdf file in Canvas in the assignment for the exam. After uploading the pdf file you notify the host again.
- Students who have obtained permission for extra time may use an *additional 30 minutes*.

Exercise 1

Consider the following LP which is referred to as the “primal LP”.

$$
\begin{aligned}
\min \quad & w = 2x_1 + 3x_2 - x_3 + x_4 \\
\text{s.t.} \quad & x_1 + 2x_2 - x_3 + 3x_4 \geq 16 \\
& 2x_1 + x_2 + x_3 + 2x_4 = 14 \\
& x_1, x_2, x_3, x_4 \geq 0
\end{aligned}
$$

(a) **[10 points]** Determine the dual of this LP.

(b) **[5 points]** The primal LP has been solved by the simplex method resulting in the following final tableau where s_1 is the surplus variable introduced in the first constraint, r_1 is the artificial variable introduced in the first constraint and r_2 is the artificial variable introduced in the second constraint.

Basic	w	x_1	x_2	x_3	x_4	s_1	r_1	r_2	value
w	1	−2.20	−2.20	0	0	−0.60	0.60	−0.40	4
x_4	0	0.60	0.60	0	1	−0.20	0.20	0.20	6
x_3	0	0.80	−0.20	1	0	0.40	−0.40	0.60	2

Determine the optimal solution of the primal LP and optimal objective value. Determine also the optimal solution of the dual LP and optimal dual objective function.

(c) **[5 points]** Suppose the right hand sides of both constraints are changed to 20 (instead of respectively 16 and 14). Determine the optimal solution and optimal objective value of this modified LP.

(d) **[10 points]** Now assume the right hand side of the first constraint is changed to 26 (instead of 16) and the right hand side of the second constraint is 14 (thus the same as in the original primal LP). Apply the dual simplex method to determine the optimal solution and optimal objective value of this modified LP.
Instruction: Since artificial variables should never become basic variables when applying the dual simplex method the columns of the two artificial variables can be omitted from the simplex tableau when you have chosen the pivot row. Moreover, it will require only one pivot step to find the new optimal solution.

Exercise 2

Bill has to buy 5 items. He can go to 4 shops which sell these items. However not every item is available in every shop. In the table you see for each shop j ($j = 1, 2, 3, 4$) which of the items i ($i = 1, 2, 3, 4, 5$) can be bought in that shop.

Shop	Items available
1	$1, 2, 3$
2	$2, 3, 4, 5$
3	$1, 3, 4, 5$
4	$1, 2, 3, 5$

In shop j the price for item i (if item i is available in shop j) is $p_{ij} > 0$. The travelling expenses for Bill to visit shop j (to buy one or more items at that shop) is $c_j > 0$. Bill wants to obtain all 5 items minimizing the total amount of prices paid and travelling expenses made.

(a) **[15 points]** Formulate Bill's problem of minimizing the total costs to obtain all 5 items as an integer linear program (ILP). Explain all variables and constraints in your ILP formulation of this problem.

Exercise 3

Consider the following ILP with four binary decision variables x_1, x_2, x_3, x_4.

$$\begin{aligned} \max \quad & z = 13x_1 + 10x_2 + 19x_3 + 25x_4 \\ \text{s.t.} \quad & 6x_1 + 5x_2 + 9x_3 + 12x_4 \leq 18 \\ & x_1, x_2, x_3, x_4 \in \{0, 1\} \end{aligned}$$

(a) **[5 points]** Write down the LP relaxation of this ILP and determine the unique optimal solution of the LP relaxation. Also determine the corresponding optimal objective value of the LP relaxation.
Instruction: The special form of the ILP makes it possible to quickly determine the optimal solution of the LP relaxation. It is not needed to apply the simplex method.

(b) **[10 points]** Solve the ILP by applying the **branch-and-bound** method. Clearly indicate in which order you compute the nodes of the search tree and where you prune the search tree. Also indicate which pruning criterion you use when you prune.

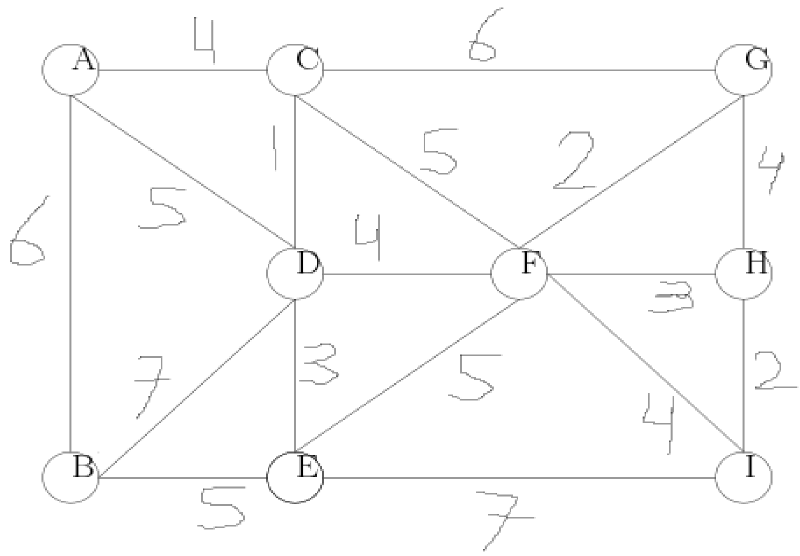

Figure 1: Exercise 4

Exercise 4

(a) **[10 points]** Consider the problem of finding a minimum weight spanning tree in the non-directed graph of Figure 1 (where edge weights have been indicated in the figure) using Kruskal's algorithm. Make clear in which **order** the edges are picked by the algorithm and draw the minimum weight spanning tree which is finally obtained. Also explicitly write down the instructions of Kruskal's algorithm which prescribe the order in which edges to be included in the minimum spanning tree are picked.

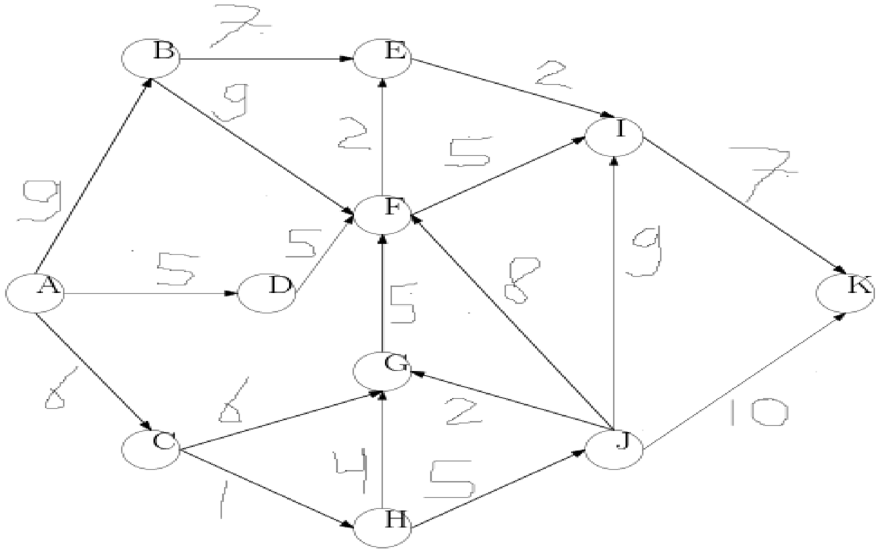

Figure 2: Exercise 5

Exercise 5

Consider the directed graph as drawn in Figure 2 with lengths of the arcs as indicated in the figure.

(a) **[10 points]** Apply Dijkstra's algorithm to determine the **shortest path** from node A to node K in this directed graph. It should be clear from the computations you write down that you have applied Dijkstra's algorithm to find the shortest path. After finishing the computations also indicate explicitly the shortest path which is obtained and the length of that shortest path.

(b) **[10 points]** Apply backward recursion to determine the **longest path** from node A to node K in the directed graph. Make sure it is clear in which order you calculate the function values and explain why the function values are calculated in that order.
Besides the function values calculated by applying backward recursion also indicate explicitly the longest path which is obtained and the length of that longest path.