VU Amsterdam	Calculus 2 for BA
Faculty of Sciences	First Test
Dr. Gabriele Benedetti	22-11-2021, 08:30-10:45

The use of a calculator, the book, or lecture notes is not permitted.
Do not just give answers, but write calculations and explain your steps.
You can score 27 points. Grade=(Points/3)+1

Question 1. (1 point, 1 point, 1 point)

Consider the sequence

$$\left\{\frac{n\sin\left(n\pi-\frac{\pi}{2}\right)}{\sqrt{2n-1}}\right\}.$$

Determine whether this sequence is

a) increasing, decreasing or alternating,

b) bounded (above and/or below),

c) convergent or divergent.

Solution.

a). Define for convenience

$$a_n := \frac{n\sin\left(n\pi-\frac{\pi}{2}\right)}{\sqrt{2n-1}}.$$

We have $\sin(1\pi-\frac{\pi}{2})=1$, while $\sin(2\pi-\frac{\pi}{2})=-1$. Since the sine function has period 2π, we conclude that $\sin\left(n\pi-\frac{\pi}{2}\right)$ is equal to 1 for n odd and to -1 for n even **(0.5 points)**. *(Alternatively:* $\sin(n\pi-\frac{\pi}{2})=-\cos(n\pi)=(-1)^{n+1}$.*)* Since the quantity $\frac{n}{\sqrt{2n-1}}$ is positive for all $n\geq 1$, a_n is alternating **(0.5 points)**.

b). We have

$$|a_n|=\frac{n}{\sqrt{2n-1}}\geq\frac{n}{\sqrt{2n}}=\sqrt{\frac{n}{2}}$$

and $\sqrt{\frac{n}{2}}$ diverges to ∞ **(0.5 points)**. Since a_n is alternating, we conclude that a_n is not bounded below nor above. **(0.5 points)**

c). Since a_n is not bounded, then a_n is divergent **(1 point)**.

Question 2. (3 points, 3 points)

Determine whether the following series are convergent or divergent. If the series is convergent, explain if it is conditionally convergent or absolutely convergent.

a) $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{3n}}{\ln(n+1)}$,

b) $\displaystyle\sum_{n=1}^{\infty}\frac{(-42)^{n}}{(n!)^2}$.

Solution.

a). We have

$$\left|\frac{(-1)^{3n}}{\ln(n+1)}\right| = \frac{1}{\ln(n+1)} \geq \frac{1}{n}. \quad \textbf{(0.5 points)}$$

As the harmonic series diverges, the given series is not absolutely convergent **(0.5 points)**. Since $(-1)^{3n} = 1$ when n is even and $(-1)^{3n} = -1$ when n is odd, the terms of the series are alternating **(0.5 points)**. Moreover, $\frac{1}{\ln(n+1)}$ is decreasing since the natural logarithm is increasing **(0.5 points)** and there holds $\lim_{n\to\infty} \frac{1}{\ln(n+1)} = 0$ **(0.5 points)**. Therefore, the series is conditionally convergent by the alternating series test. **(0.5 points)**

b). Let $a_n := \frac{(-42)^n}{(n!)^2}$. We try to apply the ratio test **(0.5 points)**: If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| < 1$, then the series converges absolutely, if $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| > 1$, then the series diverges **(0.5 points)**. We compute

$$\left|\frac{a_{n+1}}{a_n}\right| \overset{\textbf{(0.5p)}}{=} \frac{42^{n+1}}{((n+1)!)^2}\frac{(n!)^2}{42^n} = \frac{42(n!)^2}{(n+1)^2(n!)^2} \overset{\textbf{(0.5p)}}{=} \frac{42}{(n+1)^2}.$$

Taking the limit, we get

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{42}{(n+1)^2} = 0 < 1. \quad \textbf{(0.5 points)}$$

Therefore the series converges absolutely **(0.5 points)**.

Other possibility: Observe that

$$\sum_{n=1}^{\infty}|a_n| \leq \sum_{n=0}^{\infty}\frac{42^n}{n!}$$

and then use that the second series converges absolutely either applying the ratio test to it or by observing that it is equal to e^{42}.

Question 3. (4 points)

Determine the interval of convergence of the power series

$$\sum_{n=1}^{\infty}\frac{\sqrt{n}(1-3x)^n}{2^{n+1}}.$$

Solution. Using that $1 - 3x = -3(x - \frac{1}{3})$ we rewrite the series as

$$\sum_{n=1}^{\infty} a_n(x-\tfrac{1}{3})^n, \qquad a_n := \frac{\sqrt{n}(-3)^n}{2^{n+1}}, \quad \textbf{(0.5 points)}$$

therefore the center of convergence is $x = \frac{1}{3}$ **(0.5 points)**. To find the radius R of convergence, we compute first

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{\sqrt{n+1}\cdot 3^{n+1}}{2^{n+2}}\frac{2^{n+1}}{\sqrt{n}\cdot 3^n} = \sqrt{1+\tfrac{1}{n}}\cdot\frac{3}{2}.$$

Therefore,

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\sqrt{1+\tfrac{1}{n}}\cdot\frac{3}{2} = \frac{3}{2}.$$

and $R = \frac{1}{L} = \frac{2}{3}$ **(0.5 points)**. Therefore, the series converges for x in the interval $(\frac{1}{3} - \frac{2}{3}, \frac{1}{3} + \frac{2}{3}) = (-\frac{1}{3}, 1)$ **(0.5 points)**. Let's check the end-points. For $x = -\frac{1}{3}$ the series becomes

$$\sum_{n=1}^{\infty} \frac{\sqrt{n}(1 - 3(-\frac{1}{3}))^n}{2^{n+1}} = \sum_{n=1}^{\infty} \frac{\sqrt{n} \cdot 2^n}{2^{n+1}} = \sum_{n=1}^{\infty} \frac{\sqrt{n}}{2}. \quad \textbf{(0.5 points)}$$

Since $\lim_{n\to\infty} \frac{\sqrt{n}}{2} = \infty$ (and therefore this limit is different from zero), the series diverges (to ∞) **(0.5 points)**. For $x = 1$, the series becomes

$$\sum_{n=1}^{\infty} \frac{\sqrt{n}(1 - 3)^n}{2^{n+1}} = \sum_{n=1}^{\infty} \frac{\sqrt{n} \cdot (-2)^n}{2^{n+1}} = \sum_{n=1}^{\infty} (-1)^n \frac{\sqrt{n}}{2}. \quad \textbf{(0.5 points)}$$

As $\lim_{n\to\infty} \frac{\sqrt{n}}{2} = \infty$ (and thus the limit is not zero), the series diverges **(0.5 points)**. We conclude that $(-\frac{1}{3}, 1)$ is the interval of convergence.

Question 4. (2 points, 1 point)

Consider the power series

$$f(x) := \sum_{n=0}^{\infty} \frac{x^{2n+1}}{n!(2n+1)},$$

which converges for all real numbers x.

a) Show that $f'(x) = e^{x^2}$.

b) Use a) to show that $\displaystyle\int_0^1 e^{x^2}\, dx = \sum_{n=0}^{\infty} \frac{1}{n!(2n+1)}$.

Solution.
a). Differentiating the series term by term, we get

$$f'(x) \overset{\textbf{(0.5p)}}{=} \sum_{n=0}^{\infty} \frac{d}{dx}\left(\frac{x^{2n+1}}{n!(2n+1)}\right) = \sum_{n=0}^{\infty} \frac{(2n+1)x^{(2n+1)-1}}{n!(2n+1)} \overset{\textbf{(0.5p)}}{=} \sum_{n=0}^{\infty} \frac{x^{2n}}{n!}.$$

Using the MacLaurin series for the exponential

$$e^y = \sum_{n=0}^{\infty} \frac{y^n}{n!} \quad \textbf{(0.5 points)}$$

with $y = x^2$ in the above expression for $f'(x)$, we get

$$f'(x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{n!} = \sum_{n=0}^{\infty} \frac{(x^2)^n}{n!} = e^{x^2}. \qquad \textbf{(0.5 points)}$$

b). Since f is an antiderivative of e^{x^2}, the Fundamental Theorem of Calculus yields

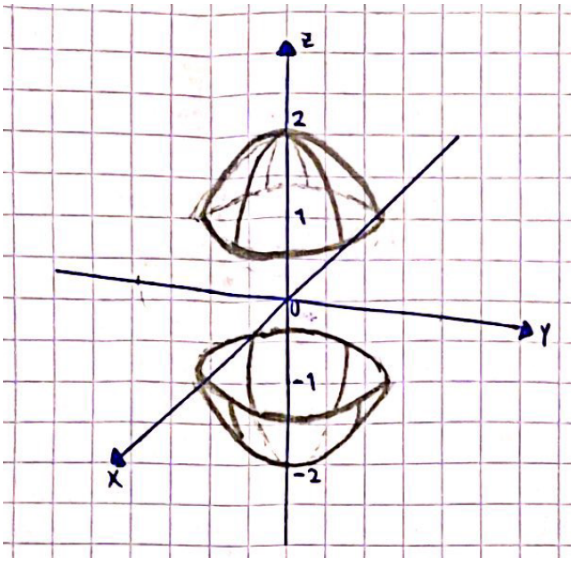
$$\int_0^1 e^{x^2}\, dx \overset{\textbf{(0.5p)}}{=} f(1) - f(0) = \sum_{n=0}^{\infty} \frac{1^{2n+1}}{n!(2n+1)} - \sum_{n=0}^{\infty} \frac{0^{2n+1}}{n!(2n+1)} \overset{\textbf{(0.5p)}}{=} \sum_{n=0}^{\infty} \frac{1}{n!(2n+1)}.$$

Question 5. (2 points)

Sketch the set of points in $\mathbb{R}^3$ that simultaneously satisfy the following conditions:

$$x^2 + y^2 + z^2 = 4, \quad |z| \geq 1.$$

Solution. The equation $x^2 + y^2 + z^2 = 4$ represents a sphere **(0.5 points)** of radius $\sqrt{4} = 2$ centered at the origin **(0.5 points)**. The points on such a sphere with $|z| \geq 1$ have either $z \geq 1$ or $z \leq -1$ **(0.5 points)** and, since $1 < 2$, they form two spherical caps around the north and south pole. **(0.5 points)**



Question 6. (1 point, 1 point, 1 point, 2 points)

The vectors **u** and **v** are given by

$$\mathbf{u} = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} = -2\mathbf{i} + \mathbf{j} + \mathbf{k}, \qquad \mathbf{v} = \begin{pmatrix} 1 \\ a \\ a^2 \end{pmatrix} = \mathbf{i} + a\mathbf{j} + a^2\mathbf{k},$$

where a is a real number.

a) Calculate the dot product $\mathbf{u} \bullet \mathbf{v}$ and the cross product $\mathbf{u} \times \mathbf{v}$ in terms of a.

b) Give all values of a for which **u** and **v** are perpendicular.

c) Give an equation for the plane that passes through the point $(1, 1, 3)$ and is normal to the vector **u**.

d) Calculate the distance from the origin to the plane from part c).

Solution.

a). We compute

$$\mathbf{u} \bullet \mathbf{v} = -2 \cdot 1 + 1 \cdot a + 1 \cdot a^2 = a^2 + a - 2. \quad \textbf{(0.5 points)}$$

Moreover,

$$\mathbf{u}\times\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 1 & 1 \\ 1 & a & a^2 \end{vmatrix} = (a^2-a)\mathbf{i} - (-2a^2-1)\mathbf{j} + (-2a-1)\mathbf{k} = \begin{pmatrix} a^2-a \\ 2a^2+1 \\ -2a-1 \end{pmatrix}. \ \textbf{(0.5 points)}$$

b). The vectors **u** and **v** are perpendicular exactly when their dot product vanishes **(0.5 points)**. This happens when $a^2+a-2=0$. Using the ABC formula we find that this equation has the solutions $a=-2$ and $a=1$ **(0.5 points)**.

c). The equation of the plane is $-2(x-1)+1(y-1)+1(z-3)=0$ **(0.5 points)** which can be rewritten as $-2x+y+z=2$ **(0.5 points)**.

d). The line through the origin which is perpendicular to the plane has parametric equation

$$x=-2t, \quad y=t, \quad z=t, \qquad t\in\mathbb{R}. \quad \textbf{(0.5 points)}$$

We find the value of t corresponding to the intersection between the line and the plane:

$$-2(-2t)+t+t=2 \iff 6t=2 \iff t=\tfrac{1}{3}.$$

The intersection point is therefore $(-\frac{2}{3},\frac{1}{3},\frac{1}{3})$ **(0.5 points)**. The distance d between the plane and the origin is the distance between this point and the origin. Therefore,

$$d \overset{\textbf{(0.5p)}}{=} \sqrt{\left(-\tfrac{2}{3}-0\right)^2+\left(\tfrac{1}{3}-0\right)^2+\left(\tfrac{1}{3}-0\right)^2} = \tfrac{1}{3}\sqrt{2^2+1^2+1^2} \overset{\textbf{(0.5p)}}{=} \frac{\sqrt{6}}{3},$$

which is also equal to $\frac{2}{\sqrt{6}}$ or $\sqrt{\frac{2}{3}}$.

Alternatively, one can compute the distance from a plane $Ax+By+Cz=D$ and a point (x_0,y_0,z_0) using the formula

$$d=\frac{|Ax_0+By_0+Cz_0-D|}{\sqrt{A^2+B^2+C^2}}, \quad \textbf{(1 point)}$$

which yields in this case

$$d=\frac{|-2\cdot 0+1\cdot 0+1\cdot 0-2|}{\sqrt{(-2)^2+1^2+1^2}}=\frac{2}{\sqrt{6}}=\frac{\sqrt{6}}{3}=\sqrt{\frac{2}{3}}. \quad \textbf{(1 point)}$$

Question 7. (2 points, 2 points)

Consider the function $f:\mathbb{R}^2\to\mathbb{R}$ given by

$$f(x,y)=y\sin\left(\frac{x}{y}\right).$$

a) Compute the partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.

b) Find the equation in standard form of the line that is normal to the graph of f at the point where $x=\pi$ and $y=1$.

Solution.

a). We calculate

$$\frac{\partial f}{\partial x} = f_1(x,y) = y\cos\left(\frac{x}{y}\right)\cdot\frac{1}{y} = \cos\left(\frac{x}{y}\right), \qquad \textbf{(1 point)}$$

$$\frac{\partial f}{\partial y} = f_2(x,y) = \sin\left(\frac{x}{y}\right) + y\cos\left(\frac{x}{y}\right)\cdot\left(-\frac{x}{y^2}\right) = \sin\left(\frac{x}{y}\right) - \frac{x}{y}\cos\left(\frac{x}{y}\right). \qquad \textbf{(1 point)}$$

b). We compute $f(\pi,1) = \sin(\pi) = 0$, $f_1(\pi,1) = \cos(\pi) = -1$, $f_2(\pi,1) = -\pi\cos(\pi) = \pi$ **(0.5 points)**. The normal line has direction vector

$$\begin{pmatrix} f_1(\pi,1) \\ f_2(\pi,1) \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ \pi \\ -1 \end{pmatrix}. \qquad \textbf{(0.5 points)}$$

Therefore, the normal line has parametric equation

$$x = \pi - t, \quad y = 1 + \pi t, \quad z = 0 - t. \qquad \textbf{(0.5 points)}$$

Eliminating t from these equations yields the standard form

$$\pi - x = \frac{y-1}{\pi} = -z. \qquad \textbf{(0.5 points)}$$

Alternatively, one can use the equation from the book for the normal line at $x = a$, $y = b$ to the graph of $f(x,y)$

$$\frac{x-a}{f_1(a,b)} = \frac{y-b}{f_2(a,b)} = \frac{z - f(a,b)}{-1}, \qquad \textbf{(1 point)}$$

which in this case yields $\pi - x = \frac{1}{\pi}(y-1) = -z$ **(1 point)**.