First test Calculus 1, 24-09-2018, Solutions.

1. Long division yields $P(x) = (x-3)(x^2+2x-9)$, so $x-3$ is clearly a (linear) factor of P. Now $P(x) = 0$ gives $x-3 = 0$ or $x^2+2x-9 = 0$. So the roots of P (the solutions of $P(x) = 0$) are $x = 3$, $x = -1+\sqrt{10}$ and $x = -1-\sqrt{10}$.

2. When we use the trigonometric identity $\sin(\pi-\theta) = \sin(\theta)$ [or use reflection of the point $(\cos(t), \sin(t))$ on the unit circle in the y-axis!] we find that $\sin(\pi-\theta) = \frac{3}{5}$. Next we use another trigonometric identity: $\sin^2(y)+\cos^2(y) = 1$. Substitute $y = \pi - x$, to find $\cos^2(\pi-x) = 1-\left(\frac{3}{5}\right)^2 = \frac{16}{25}$. Since $\theta \in [0, \frac{1}{2}\pi]$ we have $\pi-\theta \in [\frac{1}{2}\pi, \pi]$ and therefore $\cos(\pi-\theta) < 0$, so $\cos(\pi-\theta) = -\frac{4}{5}$.

3. a) The domain of f consists of all $x \neq 0$ that satisfy the inequality $5-x-\frac{4}{x} \geq 0$. To solve this inequality we move every term to the left-hand side of the inequality sign and we make one fraction:

$$5-x-\frac{4}{x} \geq 0 \Longrightarrow \frac{5x-x^2-4}{x} \geq 0 \Longrightarrow \frac{-(x-1)(x-4)}{x} \geq 0$$
$$\Longrightarrow \frac{(x-1)(x-4)}{x} \leq 0.$$

The left-hand side is 0 for $x = 1$ and $x = 4$. Furthermore it can only be negative if all three factors are negative (which is true for $x < 0$), or if one factor is negative and the other two are positive (which is true for $1 < x < 4$). So the solution set is $S = (-\infty, 0) \cup [1, 4]$. [Of course you can also organize this sign information in a chart, as is presented in Adams, section P.1.]

   b) Use the chain rule to find

$$f'(x) = \frac{-1+\frac{4}{x^2}}{2\sqrt{5-x-\frac{4}{x}}}.$$

When $f'(x) > 0$ the function f is increasing, which implies that f is increasing when $-1+\frac{4}{x^2} > 0$ as far as $x \in D_f$. This yields $-2 < x < 2$ and $x \in D_f$, so f is increasing on $[-2, 0)$ and on $[1, 2]$.

4. a) Multiply numerator and denominator by the conjugate of the expression in the denominator, to get:

$$\lim_{x\to 0} \frac{x}{\sqrt{3+\sin(x)}-\sqrt{3-\sin(x)}} \times \frac{\sqrt{3+\sin(x)}+\sqrt{3-\sin(x)}}{\sqrt{3+\sin(x)}+\sqrt{3-\sin(x)}}$$
$$= \lim_{x\to 0} \frac{x(\sqrt{3+\sin(x)}-\sqrt{3-\sin(x)})}{(3+\sin(x))-(3-\sin(x))}$$
$$= \lim_{x\to 0} \frac{1}{2} \times \frac{x}{\sin(x)} \times (\sqrt{3+\sin(x)}-\sqrt{3-\sin(x)}) = \frac{1}{2} \times 1 \times 2\sqrt{3} = \sqrt{3}.$$

b) Use the fact that for $x < 0$ we have $\sqrt{x^2} = |x| = -x$. Then we find:

$$\lim_{x\to-\infty} \frac{\sqrt{1+2x+3x^2}}{4+5x} = \lim_{x\to-\infty} \frac{\sqrt{x^2(\frac{1}{x^2}+\frac{2}{x}+3)}}{4+5x}$$

$$= \lim_{x\to-\infty} \frac{-\sqrt{\frac{1}{x^2}+\frac{2}{x}+3}}{\frac{4}{x}+5} = \frac{-\sqrt{3}}{5} = -\frac{1}{5}\sqrt{3}.$$

5. a) For continuity we must have $\lim_{x\to 0} f(x) = f(0) = b$. Now

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} a|\sin(x)| + \tan(x) = a \times 0 + 0 = 0,$$

so $b = 0$ and a can be any real number.

b) First of all f has to be continuous at $x = 0$, so $b = 0$. Then, f is differentiable at $x = 0$ if the following limit exists:

$$f'(0) = \lim_{x\to 0} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0} \frac{a|\sin(x)| + \tan(x)}{x}.$$

Distinguish between $\lim_{x\to 0+}$ and $\lim_{x\to 0-}$ (and use $\lim_{x\to 0} \frac{\tan(x)}{x} = \lim_{x\to 0} \frac{\sin(x)}{x\cos(x)} = 1$):

$$\lim_{x\to 0+} \frac{a|\sin(x)| + \tan(x)}{x} = \lim_{x\to 0+} a \times \frac{\sin(x)}{x} + \frac{\tan(x)}{x} = a + 1$$

and

$$\lim_{x\to 0-} \frac{a|\sin(x)| + \tan(x)}{x} = \lim_{x\to 0-} a \times \frac{-\sin(x)}{x} + \frac{\tan(x)}{x} = -a + 1.$$

This is only equal for $a = 0$. Therefore f is differentiable at $x = 0$ if $a = 0$ and $b = 0$ and in that case $f'(0) = 1$.

6. a) We use implicit differentiation, the product rule and the chain rule to find:

$$9x^2 - 2xy - x^2\frac{dy}{dx} + 3y^2\frac{dy}{dx} = 0, \qquad \text{which leads to} \qquad \frac{dy}{dx} = \frac{2xy - 9x^2}{3y^2 - x^2}.$$

b) The slope of the tangent line is

$$\left.\frac{dy}{dx}\right|_{(1,2)} = \left.\frac{2xy - 9x^2}{3y^2 - x^2}\right|_{(1,2)} = -\frac{5}{11}.$$

And therefore the equation of the tangent line is $y = -\frac{5}{11}(x-1)+2 = -\frac{5}{11}x + \frac{27}{11}$.

7. Choose an arbitrary $x > 0$ and consider the function $f(t) = \sqrt[3]{8+5t} = (8+5t)^{1/3}$. Then f is continuous on $[0, x]$ and differentiable on $(0, x)$. So according to the Mean Value Theorem there exists a c in $(0, x)$ so that:

$$\frac{\sqrt[3]{8+5x} - 2}{x} = \frac{f(x) - f(0)}{x - 0} = f'(c) = \frac{5}{3}(8+5c)^{-2/3} < \frac{5}{3} \cdot \frac{1}{4} = \frac{5}{12},$$

because $c > 0$, so $8 + 5c > 8$ and thus $(8+5c)^{2/3} > 8^{2/3} = 4$. Multiply both sides by x and bring 2 to the right-hand side, to obtain that $\sqrt[3]{8+5x} < 2 + \frac{5}{12}x$ for all $x > 0$.