First test Calculus 1, 26-09-2016, Solutions.

1. Move every term to the left hand side of the inequality sign and make one fraction:

$$\frac{x}{3} - 1 - \frac{2}{x+2} \leq 0 \Longrightarrow \frac{x(x+2) - 3(x+2) - 6}{3(x+2)} \leq 0$$

$$\Longrightarrow \frac{x^2 - x - 12}{3(x+2)} \leq 0 \Longrightarrow \frac{(x-4)(x+3)}{3(x+2)} \leq 0.$$

The left-hand side is 0 for $x = -3$ or $x = 4$. Furthermore it can only be negative if all three factors are negative (which is true for $x < -3$), or if one factor is negative and the other two are positive (which is true for $-2 < x < 4$). So the solution set is $S = (-\infty, -3] \cup (-2, 4]$.

2. Notice that we can restrict ourselves to $\theta \in [\pi/2, \pi]$ since $\tan(\theta)$ is negative. Now since $\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)} = -\frac{3}{4}$ we have $\sin(\theta) = -\frac{3}{4}\cos(\theta)$. Furthermore we have for all $\theta \in \mathbb{R}$ the identity $\sin^2(\theta) + \cos^2(\theta) = 1$. Combining both results yields

$$\frac{9}{16}\cos^2(\theta) + \cos^2(\theta) = 1, \qquad \text{so} \qquad \cos^2(\theta) = \frac{16}{25}.$$

Since $\theta \in [\pi/2, \pi]$ we find $\cos(\theta) = -\frac{4}{5}$ and therefore $\sin(\theta) = \frac{3}{5}$.

3. a) $\lim_{x\to 3}\left(\frac{1}{x-3} - \frac{6}{x^2-9}\right) = \lim_{x\to 3}\frac{(x+3)-6}{(x-3)(x+3)} = \lim_{x\to 3}\frac{1}{x+3} = \frac{1}{6}.$

b) Distinguish between $\lim_{x\to 0+}$ and $\lim_{x\to 0-}$ and use the standard limit $\lim_{x\to 0}\frac{\sin(x)}{x} = 1$. Then

$$\lim_{x\to 0+}\frac{|x-x^2|}{\sin(x)} = \lim_{x\to 0+}\frac{x(1-x)}{\sin(x)} = 1,$$

while

$$\lim_{x\to 0-}\frac{|x-x^2|}{\sin(x)} = \lim_{x\to 0-}\frac{-x(1-x)}{\sin(x)} = -1.$$

So the limit does not exist.

c) Multiply numerator and denominator by the conjugate of this expression:

$$\lim_{x\to\infty}\left(x - \sqrt{x^2+2x-1}\right) \times \frac{x + \sqrt{x^2+2x-1}}{x+\sqrt{x^2+2x-1}} = \lim_{x\to\infty}\frac{-2x+1}{x+\sqrt{x^2+2x-1}}$$

$$= \lim_{x\to\infty}\frac{-2+\frac{1}{x}}{1+\sqrt{1+\frac{2}{x}-\frac{1}{x^2}}} = \frac{-2}{1+1} = -1.$$

4. For continuity we must have $\lim_{x\to 0} f(x) = f(0) = 3$. But since

$$-1 \leq \sin(1/x^3) \leq 1, \qquad \text{thus} \qquad -x^2 \leq x^2 \sin(1/x^3) \leq x^2,$$

we find with the squeeze law that $\lim_{x\to 0} x^2 \sin(1/x^3) = 0 \neq f(0)$, so f has a discontinuity at $x = 0$. However, if we redefine $f(0) = 0$, f is continuous at $x = 0$, so it was a removable discontinuity.

5. If f is differentiable at $x = 0$ it must also be continuous at $x = 0$. Therefore we calculate $\lim_{x\to 0-} f(x) = b$ and $\lim_{x\to 0+} f(x) = \tan(\pi/4) = 1$. So $b = 1$. Furthermore we must have that $f'_-(0) = f'_+(0)$. We can see immediately that $f'_-(x) = 2x + a$, so that $f'_-(0) = a$. Now remark that the function $\tan(x + \pi/4)$ is differentiable for all x in a small interval containing 0. In this case we have:

$$f_+(0) = \frac{d}{dx}(\tan(x + \pi/4))\Big|_{x=0} = \frac{1}{\cos^2(x + \pi/4))}\Big|_{x=0} = \frac{1}{(\frac{1}{2}\sqrt{2})^2} = 2.$$

So $a = 2$ and $b = 1$.

6. We define $f(x) = x^5 + 4x + \cos(3x)$ which is a differentiable (thus continuous) function on $\mathbb{R}$. First we will prove that the equation $f(x) = 0$ has at least one solution. Consider the interval $[-1, 0]$. Then $f(-1) = -5 + \cos(-3) < 0$ and $f(0) = \cos(0) = 1 > 0$. Since f is continuous on $[-1, 0]$ the Intermediate Value Theorem implies that there exists a $c \in (-1, 0)$ where $f(c) = 0$.
Next we will prove that the equation $f(x) = 0$ has at most one solution. Therefore consider $f'(x) = 5x^4 + 4 - 3\sin(3x) > 0$ for all x, so f is strictly increasing on $\mathbb{R}$ and therefore we have at most one $d \in \mathbb{R}$ where $f(d) = 0$.
Combining both results proves that the equation has exactly one (real) solution.

7. We use implicit differentiation, the product rule and the chain rule to find:

$$\sin(y) + x\cos(y)\frac{dy}{dx} + 3y^2\frac{dy}{dx} = 2\cos(x) - 2x\sin(x).$$

So the slope of the tangent line is:

$$\frac{dy}{dx}\Big|_{(\pi/2,0)} = \frac{2\cos(x) - 2x\sin(x) - \sin(y)}{x\cos(y) + 3y^2}\Big|_{(\pi/2,0)} = \frac{0 - \pi - 0}{\pi/2 + 0} = -2$$

And therefore the equation of the tangent line is $y = -2(x - \pi/2) + 0 = -2x + \pi$.