First test Calculus 1, 22-09-2015, Solutions.

1. Distinguish between $2x+1<0$ and $2x+1>0$. In the first case the left-hand side of the inequality is always negative or zero (when $x=-3$), so the inequality is always true in this case. For $2x+1>0$ we also have that $|x+3|=x+3$. So, after multiplying both sides of the inequality by $2x+1$ we derive:

$$x+3<2(2x+1)=4x+2 \quad\Longrightarrow\quad 3x>1 \quad\Longrightarrow\quad x>\frac{1}{3}.$$

So the solution is $x\in(-\infty,-\frac{1}{2})\cup(\frac{1}{3},\infty)$.

2. D_f consists of all x which satisfy $x-2\geq 0$ and $1-\sqrt{x-2}\neq 0$. The last restriction gives $x\neq 3$. So $D_f=[2,3)\cup(3,\infty)$.
Now remark that $f(2)=1$ and that:

$$\lim_{x\to 3^-} f(x)=\infty, \qquad \lim_{x\to 3^+} f(x)=-\infty \qquad \text{and} \qquad \lim_{x\to\infty} f(x)=0.$$

Furthermore

$$f'(x)=\frac{1}{2(1-\sqrt{x-2})^2\sqrt{x-2}}>0 \quad \text{for all } x\in D_f\setminus\{2\},$$

so f is increasing on $[2,3)$ and also on $(3,\infty)$. Therefore $R_f=(-\infty,0)\cup[1,\infty)$.

3. a) Multiply numerator and denominator by the conjugate of the expression in the denominator:

$$\lim_{x\to 1}\frac{x^2-1}{\sqrt{x+3}-2}=\lim_{x\to 1}\frac{x^2-1}{\sqrt{x+3}-2}\times\frac{\sqrt{x+3}+2}{\sqrt{x+3}+2}=$$
$$\lim_{x\to 1}\frac{(x-1)(x+1)(\sqrt{x+3}+2)}{x-1}=\lim_{x\to 1}(x+1)(\sqrt{x+3}+2)=2(\sqrt{4}+2)=8.$$

b) We use the fact that $\sqrt{x^2}=-x$ for $x<0$:

$$\lim_{x\to-\infty}\frac{\sqrt{9x^2-5x+3}}{3x-7}=\lim_{x\to-\infty}\frac{\sqrt{x^2}\sqrt{9-\frac{5}{x}+\frac{3}{x^2}}}{x(3-\frac{7}{x})}$$
$$=\lim_{x\to-\infty}\frac{-\sqrt{9-\frac{5}{x}+\frac{3}{x^2}}}{3-\frac{7}{x}}=\frac{-\sqrt{9}}{3}=-1.$$

c) We use the fact that $\tan t=\frac{\sin t}{\cos t}$ and the standard limit $\lim\limits_{t\to 0}\frac{\sin t}{t}=1$:

$$\lim_{x\to 0}\frac{2x-\sin(3x)}{\tan(x)+4x}=\lim_{x\to 0}\frac{2-\frac{\sin(3x)}{3x}\cdot 3}{\frac{\sin(x)}{x}\cdot\frac{1}{\cos(x)}+4}=\frac{2-1\cdot 3}{1\cdot 1+4}=-\frac{1}{5}$$

4. For continuity we must have:

$$\lim_{x\to\pi} f(x) = f(\pi) = \cos\left(\frac{3}{4}\pi\right) = -\frac{1}{2}\sqrt{2}.$$

We calculate the one-side limits:

$$\lim_{x\to\pi-} f(x) = \pi + k \quad \text{and} \quad \lim_{x\to\pi+} f(x) = \cos\left(\frac{3}{4}\pi\right) = -\frac{1}{2}\sqrt{2}.$$

So if we choose $k = -\pi - \frac{1}{2}\sqrt{2}$ the function is continuous at $x = \pi$.

5. If $f'(0)$ exists it must be equal to:

$$\lim_{h\to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0} \frac{h^2 \sin\left(\frac{1}{h}\right) - 0}{h} = \lim_{h\to 0} h \sin\left(\frac{1}{h}\right).$$

The last limit equals 0 and can be calculated using the squeeze theorem. Since

$$-|h| \leq h \sin\left(\frac{1}{h}\right) \leq |h|$$

and since $\pm|h|$ tend to 0 if h tends to 0, we find $f'(0) = 0$.

6. We use implicit differentiation, the product rule and the chain rule to find:

$$y + x \cdot \frac{dy}{dx} + 3y^2 \cdot \frac{dy}{dx} = 4x.$$

So the slope of the tangent line is:

$$\left.\frac{dy}{dx}\right|_{(1,1)} = \left.\frac{4x - y}{x + 3y^2}\right|_{(1,1)} = \frac{3}{4}$$

And therefore the equation of the tangent line is $y = \frac{3}{4}(x-1) + 1 = \frac{3}{4}x + \frac{1}{4}$.

7. Choose $x \in (0, \frac{\pi}{4})$ arbitrarily and define $f(t) = \tan(t)$. Then f is continuous on $[0, x]$ and differentiable on $(0, x)$. So according to the Mean Value Theorem there exists a c in $(0, x)$ such that:

$$\frac{\tan(x)}{x} = \frac{f(x) - f(0)}{x - 0} = f'(c) = \frac{1}{\cos^2(c)} < 2,$$

because on $(0, \frac{1}{4}\pi)$ we have

$$\frac{1}{2}\sqrt{2} < \cos(c) < 1, \text{ thus } \frac{1}{2} < \cos^2(c) < 1 \text{ and } 1 < \frac{1}{\cos^2(c)} < 2.$$

It follows that $\tan(x) < 2x$ on $(0, \frac{\pi}{4})$.