Investments 4.1
Course code: 60412040

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Sample Exam!

Parts:	This sample exam contains 20 parts. In a real exam, each part would yield roughly the same number of points.
Grading:	(Does not apply)
Results:	(Does not apply)
Inspection:	(Does not apply)
Remark:	**Be complete, but concise!** Provide complete answers, including computations where appropriate. On verbal questions: always provide motivation/explanation of your answer in terms of the economic mechanism at play. A short “yes” or “no” will never do as an answer. But be concise in your answers, otherwise you’ll loose to much time writing it down.

Scan for the questions you find easiest and solve them first!

Useful formulas

$$\frac{\partial x'a}{\partial a} = \frac{\partial a'x}{\partial a} = x \quad (1)$$

$$\frac{\partial x'Ax}{\partial x} = (A + A')x \quad (2)$$

$$\text{Var}(x) = E[x^2] - E[x]^2 \quad (3)$$

$$\text{Cov}(x, y) = E[x \cdot y] - E[x]E[y] \quad (4)$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \quad (5)$$

1. An investor who is a mean-variance optimizer wants to allocate her portfolio optimally between two stocks and risk free bonds. The returns to the two stocks are jointly normally distributed with $R_{t+1} \sim \mathcal{N}(\mu, \Omega)$. The investment opportunity set faced by investor is characterized by

$$R_f = 0.05 \qquad \mu = \begin{bmatrix} 0.15 \\ 0.1 \end{bmatrix} \qquad \Omega = \begin{bmatrix} 1/9 & 1/27 \\ 1/27 & 1/9 \end{bmatrix}$$

The investor chooses a vector of portfolio weights w for the two stocks to maximize:

$$w'\mu + (1 - w'\iota)R_f - \frac{\lambda}{2}w'\Omega w$$

(a) Show (using matrix algebra) that the optimal w is given by:

$$w = \frac{1}{\lambda}\Omega^{-1}(\mu - R_f\iota)$$

FOC: Derivative with respect to $w = 0$:

$$\mu - \iota R_f - \lambda\Omega w = 0 \Rightarrow w = \frac{1}{\lambda}\Omega^{-1}(\mu - R_f\iota)$$

(b) Compute the optimal w for an investor with $\lambda = 2$. Explain why the investor allocates more to one of the stocks than the other.

$$w = \frac{1}{2}\begin{bmatrix} 10.125 & -3.3750 \\ -3.3750 & 10.1250 \end{bmatrix}\begin{bmatrix} 0.1 \\ 0.05 \end{bmatrix} = \begin{bmatrix} 42.2\% \\ 8.4 \end{bmatrix} \quad (6)$$

The optimal portfolio has a higher weight in the first stock, because it has a higher expected return and the same variance as the second one.

(c) Compute the expected return and variance of the investor's portfolio.

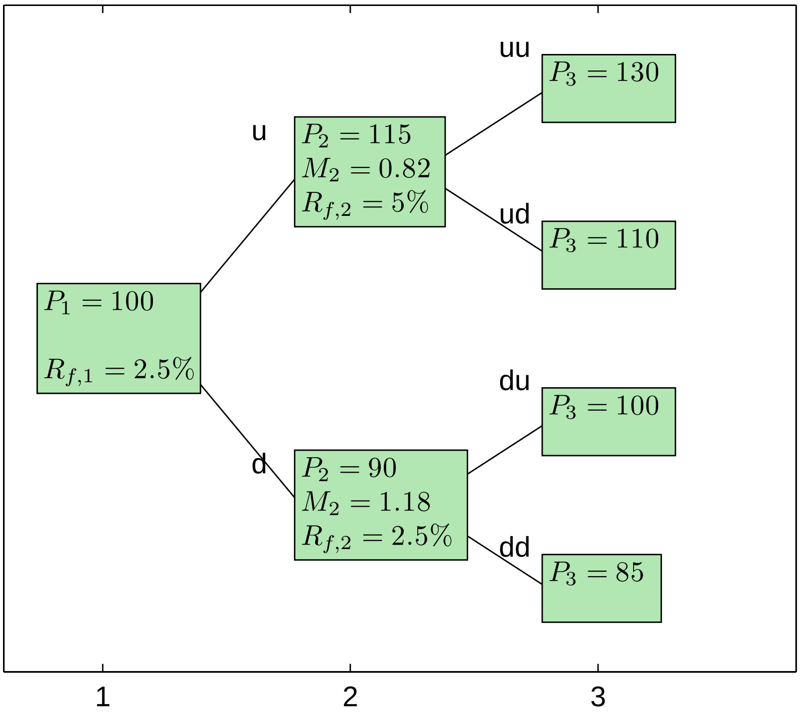
$$E[R_p] = w'\mu + (1 - w'\iota)R_f = 0.0964$$
$$\text{Var}(R_p) = w'\Omega w = 0.0232$$

(d) Find the mean and variance of the market portfolio.

$$E[R_m] = 0.1417$$
$$\text{Var}(R_m) = 0.0905$$

(e) Sketch a figure with the Efficient frontier and the capital markets line and place the investor's portfolio as well as the market portfolio in the figure. Remember to label the x-axis and the y-axis.

2. Consider the following simple economy. The probability of going up and down is equal to 50% always. P_t is the price of a stock in period t and M_{t+1} is the value of the stochastic discount factor between times t and $t+1$ depending on the state of the economy. The nodes of the tree are named according to whether we went up (u) or down (d).



(a) Using the Euler equation for equity and risk-free bonds, find the stochastic discount factors at each of the period 3 nodes (uu, ud, du, and dd).

Solution for uu and ud:

$$1.05^{-1} = 0.5M_{uu} + 0.5M_{ud}$$
$$1 = 0.5\frac{130}{115}M_{uu} + 0.5\frac{110}{115}M_{ud}$$

Solve either by matrix algebra or by eliminating variables, etc. This gives:

$$\begin{bmatrix} M_{uu} \\ M_{ud} \end{bmatrix} \begin{bmatrix} 1.0238 \\ 0.8810 \end{bmatrix}$$

Same calculations for M_{du} and M_{dd} yields

$$\begin{bmatrix} M_{du} \\ M_{dd} \end{bmatrix} \begin{bmatrix} 0.9431 \\ 1.0081 \end{bmatrix}$$

(b) Arrow-Debreu securities are securities that pay out 1 unit if a particular state is realized and zero otherwise. Compute the price at time 2 and at time 1 of the four Arrow-Debreu securities that pay out 1 unit in each of the four possible final states. If you were not able to solve the first question, use the following 4 stochastic discount factors at uu, ud, du, and dd: [1 0.9 1.2 0.5]

In the second period:

$$\begin{bmatrix} p_{uu}|u \\ p_{ud}|u \\ p_{du}|u \\ p_{dd}|u \end{bmatrix} = 0.5 \begin{bmatrix} M_{uu} \\ M_{ud} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.5119 \\ 0.4405 \\ 0 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} p_{uu}|d \\ p_{ud}|d \\ p_{du}|d \\ p_{dd}|d \end{bmatrix} = 0.5 \begin{bmatrix} 0 \\ 0 \\ M_{du} \\ M_{dd} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0.4715 \\ 0.5041 \end{bmatrix}$$

In the first period:

$$\begin{bmatrix} p_{uu} \\ p_{ud} \\ p_{du} \\ p_{uu} \end{bmatrix} = 0.5M_u \begin{bmatrix} p_{uu}|u \\ p_{ud}|u \\ p_{du}|u \\ p_{dd}|u \end{bmatrix} + 0.5M_d \begin{bmatrix} p_{uu}|d \\ p_{ud}|d \\ p_{du}|d \\ p_{dd}|d \end{bmatrix} = \begin{bmatrix} 0.2099 \\ 0.1806 \\ 0.2782 \\ 0.2974 \end{bmatrix}$$

(c) $\zeta(t)$ is the price at time t of a security that pays zero if the stock price is below 100 at time 3 and $(P_3 - 100)$ if the price of the stock is above 100 at time 3 and nothing in between.[1] Find $\zeta(2)$ at each of the two nodes and $\zeta(1)$. Again, if you didn't solve question (a) use the same stochastic discount factors as in (b)

$$\zeta(2)|u = 0.5 \cdot M_{uu}30 + 0.5 \cdot M_{ud}10 = 19.76$$
$$\zeta(2)|d = 0$$

$$\zeta(1) = 0.5 \cdot M_u \cdot (zeta(2)|u) = 8.10$$

3. In class we showed that the fundamental asset pricing equation implies the relation:

$$\frac{E[R_i] - R_f}{\sigma_i} = -\rho\frac{\sigma_M}{E[M]}$$

Where R_i is the the return to asset i, σ_i it's standard deviation, ρ the correlation between returns to asset i and the stochastic discount factor and σ_M and $E[M]$ are the standard deviation and mean of the stochastic discount factor, respectively.

(a) Show that this equality implies the inequality

$$(SR_i)^2 \leq \frac{\sigma_M^2}{E[M]^2}$$

where SR_i is the Sharpe ratio on asset i.

Squaring both sides of the equality:

$$(SR_i)^2 = \rho^2\frac{\sigma_M^2}{E[M]^2}$$

Because $-1 < \rho < 1$, $\rho^2 < 1$, so

$$\rho^2\frac{\sigma_M^2}{E[M]^2} \leq \frac{\sigma_M^2}{E[M]^2} \tag{7}$$

The inequality follows immediately.

(b) Suppose that you are interested in the linear asset pricing model:

$$M_{t+1} = 0.99 - \gamma\Delta c_{t+1}$$

Where the $\Delta c_{t+1} \sim N(0.02, 0.02^2)$. From stock market data, your estimate of the Sharpe ratio for the market portfolio is 0.25. Compute the terms in the Hansen-Jagannathan bound for the following values of γ: 5, 10, 15.

$(SR_i)^2 = 0.0625$. Using that $\sigma_M^2 = \gamma^2 0.02^2$ and $E[M] = 0.99 - \gamma 0.02$:

$$\left.\frac{\sigma_M^2}{E[M]^2}\right|_{\gamma=5} = 0.0123$$
$$\left.\frac{\sigma_M^2}{E[M]^2}\right|_{\gamma=10} = 0.0625$$
$$\left.\frac{\sigma_M^2}{E[M]^2}\right|_{\gamma=15} = 0.1837$$

[1] This is a European call with strike 100 which expires at $t = 3$.

(c) Is the Hansen-Jagannathan bound violated for any of the values of γ you tried? Given only this evidence, which γ would you use in your asset pricing model? Explain why.

It is violated for $\gamma = 5$. $\gamma = 15$ seems to be the best value, because it's safely within the HJ-bound.

4. (a) Consider the following utility function

$$U(C) = -\frac{e^{-bC}}{b}$$

What is the coefficient of relative and absolute risk aversion?

$$U'(C) = e^{-bC}$$
$$U''(C) = -be^{-bC}$$

So

$$R_A(C) = b$$
$$R_R(C) = bC$$

(b) Consider a general utility function $U(C)$ with the standard properties[2] and assume you want to maximize

$$U(C_0) + \theta E_0[U(C_1)]$$

by changing consumption now (C_0) and w, the allocation to equity. You earn the risk free rate R_f on the part of your wealth kept in risk-free bonds, while R_1 gives the return to equity. At time 1, you consume all your wealth, implying

$$C_1 = (W_0 - C_0)(1 + (1 - w)R_f + wR_1).$$

Use the first order condition of the maximization problem with respect to w to show that the expectation of excess returns over the risk free rate $(R_1 - R_f)$ weighted by marginal utility at time 1 equals zero.

The first order condition for an optimal w:

$$0 = E_0[\frac{\partial U(C_1)}{\partial C_1}\frac{\partial C_1}{\partial w}] = E_0[U'(C_1)(R_1 - R_f)]$$

(c) Rewrite the expression $E[U'(C_1)(R_1 - R_f)] = 0$ into an expression for expected excess returns $E[(R_1 - R_f]$.[3] Also translate this expression into one involving the stochastic discount factor $M_1 = \theta U'(C_1)/U'(C_0)$.

$$0 = E_0[U'(C_1)(R_1 - R_f)] = \text{Cov}\,(U'(C_1), (R_1 - R_f)) + E_0[U'(C_1)]E_0[(R_1 - R_f)]$$

$$\Rightarrow E_0[(R_1 - R_f)] = -\frac{\text{Cov}\,(U'(C_1), (R_1 - R_f))}{E_0[U'(C_1)]}$$

Multiplying by θ and dividing by $U'(C_0)$ in both denominator and nominator:

$$\Rightarrow E_0[(R_1 - R_f)] = -\frac{\text{Cov}\,(M_1, (R_1 - R_f))}{E_0[M_1]}$$

[2] i.e. it's differentiable, marginal utility is always positive, but decreases in consumption

[3] You can use one of the equations at the start of the exam.

(d) Explain in words why equilibrium returns on an asset are higher if they covary negatively with marginal utility.

Payoffs in states where marginal utility is high are more valuable to you. For a given expected return, an asset that pays off more in states where marginal utility is low than in states where marginal utility is high will be less attractive than assets which pays off more in states where marginal utility is high and less in states where marginal utility is low. Investors need to be compensated for holding assets with unattractive payoff profiles, so their expected returns need to be higher in equilibrium.[4]

5. Assume that the representative investor is a log utility maximizer with period utility function $U(C) = \log C$ and time discount factor is 0.998. The economy can be in either of two states, a boom or a recession. Quarterly consumption growth is conditionally log-normal, with

$$\Delta c_{t+1} = \mu_{t+1} + \epsilon_{t+1}, \qquad \epsilon_{t+1} \sim \mathcal{N}(0, 0.0061^2) \qquad \mu_{t+1} = \begin{cases} 0.0075, & \text{if } s_{t+1} = 1 \\ -0.005, & \text{if } s_{t+1} = 2 \end{cases}$$

The probability of going from a boom to a recession between two quarters is 0.063, that of going from a recession to a boom is 0.2308.

(a) Compute

$$\bar{M}_1 = E[M_{t+1}|s_{t+1} = 1] \text{ and}$$
$$\bar{M}_2 = E[M_{t+1}|s_{t+1} = 2],$$

the expectation of the stochastic discount factor between time t and $t+1$, conditional on ending in state 1 or 2.

$$\bar{M}_1 = 0.9907$$
$$\bar{M}_2 = 1.0030$$

(b) Use this result as well as the state transition probabilities to compute R_1 and R_2, the one period interest rate in state 1 and 2.

$$\begin{bmatrix} R_1 \\ R_2 \end{bmatrix} = \begin{bmatrix} ((1-p)\overline{M}_1 + p\overline{M}_2)^{-1} - 1 \\ (q\overline{M}_1 + (1-q)\overline{M}_2)^{-1} - 1 \end{bmatrix} = \begin{bmatrix} 0.0086 \\ -0.0002 \end{bmatrix}$$

(c) Explain (in terms of economics) why the interest rate is higher in one of the states than in the other.

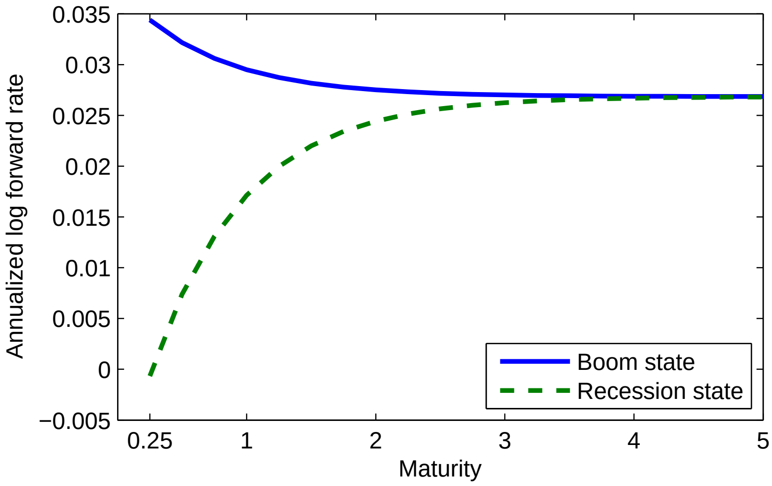
Expected consumption growth in the boom state is higher than in the recession state. Because investors like to have consumption profiles that are smooth over time, they would like to borrow more in the boom state. This will drive up interest rates in booms and depress interest rates in recessions.

(d) Explain how you would compute interest rates for other horizons.

See week 6, slide 13

(e) The following figure shows forward rates for different maturities in a boom and in a recession. (Annualized by multiplying by 4.

[4] There is a 99 % probability that you'll get a question that is similar to this one on the exam.



Sketch the corresponding spot curves and explain why they have the shape they do.

See week 6, slide 15. They have the shape they do, because the spot rate for a certain maturity is a geometric average of the forward rates between now and that maturity.